Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 170 (2016) 272–279
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

Lifts. On tensor structures

Gocha Todua

Georgian Technical University, Department of Mathematics, United States

Available online 7 May 2016

Abstract

The lifts for vector and tensor fields are constructed, some internal tensor structures are considered, the Nijenhuis tensor is found, integrability of these structures is studied, the F-structures are also considered.

Keywords: Vector fibration; Linear connectedness; Lifts; Groups; Tensor structures; Nijenhuis tensor

Let us consider a vector fiber space Lm(Vn) whose local coordinates of the point are transformed by the law [1]

$$\overline{x^i} = \overline{x^i}(x^k); \qquad \overline{y^\alpha} = A^\alpha_\beta(x) y^\beta;$$
$$\det \left\| \frac{\partial x^i}{\partial x^k} \right\| \neq 0; \qquad \det \left\| A^\alpha_\beta \right\| \neq 0; \quad i, j, k = 1, \dots, n; \;\; \alpha, \beta, \gamma = 1, \dots, m. \tag{1}$$

1. Lifts of the vector and tensor fields

Since the local coordinates (x^i, y^α) of a point of vector fibration Lm(Vn) are transformed by formulas (1), we obtain that the first differential group $\mathrm{Lm}(n, m, R)$ of the vector fibration Lm(Vn) is defined by the matrices of the type

$$\mathcal{L}^A_B = \left\| \frac{\partial \overline{\mathcal{L}}^A}{\partial \mathcal{L}^B} \right\| = \left\| \begin{matrix} \dfrac{\partial \overline{x}^i}{\partial x^j} & \dfrac{\partial \overline{x}^i}{\partial x^\alpha} \\ \dfrac{\partial \overline{x}^\alpha}{\partial x^i} & \dfrac{\partial \overline{x}^\alpha}{\partial x^\beta} \end{matrix} \right\| = \left\| \begin{matrix} x^i_j & 0 \\ A^\alpha_{\beta k} y^\beta & A^\alpha_\beta \end{matrix} \right\|.$$

E-mail address: gochatodua@yahoo.com.
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2016.04.003

The matrix, inverse to the above one, is of the form

$$\overset{*}{\mathcal{L}} = \left\| \frac{\partial \mathcal{L}^A}{\partial \overline{\mathcal{L}}^B} \right\| = \left\| \begin{matrix} \dfrac{\partial x^i}{\partial \overline{x}^k} & \dfrac{\partial x^i}{\partial \overline{x}^\alpha} \\ \dfrac{\partial x^\alpha}{\partial \overline{x}^i} & \dfrac{\partial x^\alpha}{\partial \overline{x}^\beta} \end{matrix} \right\| = \left\| \begin{matrix} \overset{*}{x}{}^i_j & 0 \\ \overset{*}{A}{}^\alpha_{\beta k} A^\beta_\gamma y^\gamma & \overset{*}{A}{}^\alpha_\beta \end{matrix} \right\|.$$

Obviously, the first differential group GL(n, m, R) has always two subgroups GL(n, R) and GL(m, R). This implies that with the vector fiber space Lm(Vn) are always connected to five sets of fields of differential geometric objects:

(1) A set of differential geometric objects $\mathfrak{J}$(Vn) (with respect to the group GL(n, R)), whose components are the functions of the point of the base Vn.

(2) A set of differential geometric objects $\mathfrak{J}$(Lm(Vn)) (with respect to the group GL(n, R)), whose components are the functions of the point of vector fibration Lm(Vn).

(3) A set of differential geometric objects $\overset{*}{\mathfrak{J}}$(Vn) (with respect to the group GL(m, R)), whose components are the functions of the point of the base Vn.

(4) A set of differential geometric objects $\overset{*}{\mathfrak{J}}$(Lm(Vn)) (with respect to the group GL(m, R)), whose components are the functions of the point of vector fibration Lm(Vn).

(5) A set of differential geometric objects $\check{\mathfrak{J}}$ Lm(Vn) (with respect to the group GL(n, m, R)), whose components are the functions of the point of vector fibration Lm(Vn). In the capacity of subsets, these sets have differential geometric objects of the first order. Moreover, among the above-mentioned differential geometric objects we distinguish five graded algebras

$$\mathfrak{J}(\mathrm{Vn}) = \sum_{p,q=0}^{\infty} \mathfrak{J}^p_q(\mathrm{Vn}), \qquad \mathfrak{J}(\mathrm{Lm}(\mathrm{Vn})) = \sum_{p,q=0}^{\infty} \mathfrak{J}^p_q(\mathrm{Lm}(\mathrm{Vn})), \qquad \overset{*}{\mathfrak{J}}(\mathrm{Vn}) = \sum_{p,q=0}^{\infty} \overset{*}{\mathfrak{J}}{}^p_q(\mathrm{Vn}),$$

$$\overset{*}{\mathfrak{J}}(\mathrm{Lm}(\mathrm{Vn})) = \sum_{p,q=0}^{\infty} \overset{*}{\mathfrak{J}}{}^p_q(\mathrm{Lm}(\mathrm{Vn})), \qquad \check{\mathfrak{J}}(\mathrm{Lm}(\mathrm{Vn})) = \sum_{p,q=0}^{\infty} \check{\mathfrak{J}}{}^p_q(\mathrm{Lm}(\mathrm{Vn})).$$

Note that for the tangent vector fibrations, certain graded algebras coincide.

If $T \in \mathfrak{J}^1_1$(Lm(Vn)), then the law of transformation of that tensor components has the form

$$\mathcal{L}^D_B \overline{T^A_D} = \mathcal{L}^A_C T^C_B, \quad A, B, C = 1, 2, \ldots, n + m. \tag{2}$$

To the tensor T there corresponds the matrix

$$\| T^A_B \| = \left\| \begin{matrix} T^i_j & T^i_\alpha \\ T^\alpha_i & T^\alpha_\beta \end{matrix} \right\|.$$

Obviously, by virtue of (1), we can write formulas (2) as follows:

$$x^p_j \overline{T^i_p} = x^i_p T^p_j - A^\alpha_{\beta j} y^\beta \overline{T^i_\alpha}, \qquad A^\beta_\alpha \overline{T^i_\beta} = x^i_p T^p_\alpha,$$
$$A^\gamma_\beta \overline{T^\alpha_\gamma} = A^\alpha_{\gamma p} y^\gamma T^p_\beta + A^\alpha_\gamma T^\gamma_\beta, \qquad x^k_i \overline{T^\alpha_k} + A^\beta_{\gamma i} y^\gamma \overline{T^\alpha_\beta} = A^\alpha_{\gamma k} y^\gamma T^k_i + A^\alpha_\beta T^\beta_i.$$

Thus we can see that the tensor T^A_B has a number of subobjects among which there are the tensor T^i_α (as an element of the algebra $\mathfrak{J}^1_1$(Lm(Vn))) and the linear homogeneous subobjects (T^i_j, T^i_α), $(T^i_\alpha, T^\alpha_\beta)$. A set of all tensors $T \in \mathfrak{J}^1_1$(Lm(Vn)), for which the values T^i_α are equal to zero, form a new subalgebra of the algebra $\mathfrak{J}^1_1$(Lm(Vn)) which we call a triangular subalgebra and denote it by $\overset{*}{\mathfrak{J}}{}^1_1$(Lm(Vn)). Then to the tensors $T \in \overset{*}{\mathfrak{J}}{}^1_1$(Lm(Vn)) there correspond the matrices of the form

$$\| T^A_B \| = \left\| \begin{matrix} T^i_j & 0 \\ T^\alpha_i & T^\alpha_\beta \end{matrix} \right\|,$$

and the values T^i_j and T^α_β form tensors.

Analogously, if $T \in \mathfrak{J}_2^1(\mathrm{Lm(Vn)})$, we have

$$\overline{T^A_{BC}} = \overset{*}{\mathcal{L}}{}^D_B \, \overset{*}{\mathcal{L}}{}^E_C \, \mathcal{L}^A_P \, T^P_{DE}.$$

Hence, owing to (1), we obtain

$$\begin{aligned}
T^i_{jk} &= \overset{*}{x}{}^q_j \, \overset{*}{x}{}^p_k \, x^i_l \, T^l_{qp} + x^q_j \, x^i_p \, \overset{*}{A}{}^\alpha_{\gamma k} \, A^\gamma_\delta \, y^\delta \, T^p_{q\alpha} + \overset{*}{x}{}^p_k \, x^i_q \, \overset{*}{A}{}^\alpha_{\gamma j} \, A^\gamma_\delta \, y^\delta \, T^q_{\alpha p} + x^i_p \, \overset{*}{A}{}^\alpha_{\gamma j} \, A^\gamma_\delta \, y^\delta \, \overset{*}{A}{}^\beta_{\sigma k} \, A^\sigma_\varepsilon \, y^\varepsilon \, T^p_{\alpha\beta}, \\
T^i_{j\alpha} &= \overset{*}{x}{}^k_j \, \overset{*}{A}{}^\beta_\alpha \, x^i_p \, T^p_{k\beta} + x^i_p \, \overset{*}{A}{}^\gamma_\alpha \, \overset{*}{A}{}^\beta_{\gamma j} \, A^\sigma_\delta \, y^\delta \, T^p_{\beta\sigma}, \qquad T^i_{\alpha\beta} = \overset{*}{A}{}^\gamma_\alpha \, \overset{*}{A}{}^\delta_\beta \, x^i_k \, T^k_{\gamma\delta}, \\
T^i_{\alpha j} &= \overset{*}{A}{}^\beta_\alpha \, \overset{*}{x}{}^p_j \, x^i_k \, T^k_{\beta p} + \overset{*}{A}{}^\beta_\alpha \, x^i_k \, \overset{*}{A}{}^\gamma_{\delta j} \, A^\delta_\sigma \, y^\sigma \, T^k_{\beta\gamma}, \qquad T^\alpha_{\beta\gamma} = \overset{*}{A}{}^\delta_\beta \, \overset{*}{A}{}^\sigma_\gamma \, A^\alpha_\varepsilon \, T^\varepsilon_{\delta\sigma} + \overset{*}{A}{}^\delta_\beta \, \overset{*}{A}{}^\sigma_\gamma \, A^\alpha_{\varepsilon i} \, y^\varepsilon \, T^i_{\delta\sigma}, \\
T^\alpha_{\beta i} &= \overset{*}{A}{}^\gamma_\beta \, \overset{*}{x}{}^p_i \, A^\alpha_{\sigma k} \, y^\sigma \, T^k_{\gamma p} + \overset{*}{A}{}^\gamma_\beta \, \overset{*}{x}{}^p_i \, A^\alpha_\varepsilon \, T^\varepsilon_{\gamma p} + \overset{*}{A}{}^\gamma_\beta \, \overset{*}{A}{}^\delta_{pi} \, y^\sigma \, A^p_\sigma \, A^\alpha_\varepsilon \, T^\varepsilon_{\gamma\delta} + \overset{*}{A}{}^\gamma_\beta \, \overset{*}{A}{}^\delta_{\sigma i} \, A^\sigma_p \, y^p \, A^\alpha_{\varepsilon k} \, y^\varepsilon \, T^k_{\gamma\delta}, \\
T^\alpha_{i\beta} &= \overset{*}{x}{}^k_i \, \overset{*}{A}{}^\gamma_\beta \, A^\alpha_{\sigma p} \, y^\sigma \, T^p_{k\gamma} + \overset{*}{x}{}^k_i \, \overset{*}{A}{}^\gamma_\beta \, A^\alpha_\delta \, T^\delta_{k\gamma} + \overset{*}{A}{}^\gamma_\beta \, \overset{*}{A}{}^\delta_{\sigma i} \, A^\sigma_\varepsilon \, y^\varepsilon \, A^\alpha_{pp} \, T^p_{\delta\gamma} + \overset{*}{A}{}^\gamma_\beta \, A^\alpha_\sigma \, \overset{*}{A}{}^\delta_{\varepsilon i} \, A^\varepsilon_p \, y^p \, T^\sigma_{\delta\gamma}, \\
T^\alpha_{ij} &= \overset{*}{x}{}^k_i \, \overset{*}{x}{}^p_j \, A^\alpha_{\beta q} \, y^\beta \, T^q_{kp} + \overset{*}{x}{}^k_i \, \overset{*}{x}{}^p_j \, A^\alpha_\beta \, T^\beta_{kp} + \overset{*}{x}{}^k_i \, \overset{*}{A}{}^\beta_{\gamma j} \, A^\gamma_\delta \, y^\delta \, A^\alpha_{\varepsilon p} \, y^\varepsilon \, T^p_{k\beta} + \overset{*}{x}{}^k_i \, A^\alpha_\gamma \, \overset{*}{A}{}^\beta_{\delta j} \, A^\delta_\sigma \, y^\sigma \, T^\gamma_{k\beta} \\
&\quad + \overset{*}{x}{}^p_j \, \overset{*}{A}{}^\beta_{\gamma i} \, A^\gamma_\delta \, y^\delta \, A^\alpha_{\varepsilon q} \, y^\varepsilon \, T^q_{\beta p} + \overset{*}{x}{}^p_j \, A^\alpha_\gamma \, \overset{*}{A}{}^\beta_{\delta i} \, A^\delta_\sigma \, y^\sigma \, T^\gamma_{\beta p} + \overset{*}{A}{}^\beta_{\delta i} \, A^\delta_\sigma \, y^\sigma \, \overset{*}{A}{}^\gamma_{\varepsilon j} \, A^\varepsilon_p \, y^p \, A^\alpha_{\omega k} \, y^\omega \, T^k_{\beta\gamma} \\
&\quad + A^\alpha_\delta \, \overset{*}{A}{}^\beta_{pi} \, A^p_\sigma \, y^\sigma \, \overset{*}{A}{}^\gamma_{\varepsilon j} \, A^\varepsilon_\omega \, y^\omega \, T^\delta_{\beta\gamma}.
\end{aligned}$$

It is not difficult to see that the tensor T^P_{DE} has a number of linear and homogeneous subobjects among which there are the tensor $T^i_{\alpha\beta}$ and the following linear homogeneous subobjects

$$(T^i_{j\alpha}, T^i_{\alpha\beta}), (T^i_{\alpha j}, T^i_{\alpha\beta}), (T^i_{\alpha\beta}, T^\alpha_{\beta\gamma}), (T^i_{jk}, T^i_{j\alpha}, T^i_{\alpha j}, T^i_{\alpha\beta}), (T^i_{\alpha k}, T^i_{\alpha\beta}, T^\alpha_{\beta i}, T^\alpha_{\beta\gamma}), (T^i_{k\alpha}, T^i_{\alpha\beta}, T^\alpha_{k\beta}, T^\alpha_{\beta\gamma}).$$

If components of the tensor $T^i_{\alpha\beta}$ are equal to zero, then the values, respectively, $T^i_{j\alpha}, T^i_{\alpha j}, T^\alpha_{\beta\gamma}$ form tensors (as elements of the algebra $\mathfrak{J}_2^1(\mathrm{Lm(Vn)})$).

If the $\{e_i, e_\alpha\}$-frame of the tangent space T_{n+m} at the point $z = (x, y) \in \mathrm{Lm(Vn)}$, then the vectors $E_i = e_i - \Gamma^\alpha_i e_\alpha$ determine an invariant equipment of the tangent space. The first differential group $GL(n, m, R)$ has always two differential subgroups $GL(n, R)$ and $GL(m, R)$. This implies that on the fiber space Lm(Vn) there exist tensor algebras with respect to the tensor products of the groups $GL(n, R)$, $GL(m, R)$, $GL(n, m, R)$. If on Lm(Vn) is assigned an object of linear connectedness, then every $GL(n, m, R)$-vector field is uniquely expanded into two vector fields with respect to the groups $GL(n, R)$ and $GL(m, R)$. Obviously, to an arbitrary field defined on the base Vn of the space Lm(Vn) there always corresponds the $GL(n, m, R)$-vector field defined on the whole fiber space. Analogous correspondence takes also place between tensors of another valencies.

Let ξ^A be the $GL(n, m, R)$-vector field defined on Lm(Vn), i.e.,

$$d\xi^A + \xi^B \, \omega^A_B = \xi^A_k \, \omega^k + \xi^A_\alpha \, \widetilde{\theta}^\alpha, \quad A, B, C = 1, 2, \ldots, n+m.$$

Then

$$\xi = \xi^A \, e_A = \xi^i \, e_i + \xi^\alpha \, e_\alpha = \xi^i \, E_i + (\xi^\alpha + \Gamma^\alpha_k \, \xi^k) e_\alpha.$$

Definition. The vector field ξ^i is called a horizontal projection of the $GL(n, m, R)$-vector field, and the vector field $\xi^\alpha + \Gamma^\alpha_k \, \xi^k$ is called a vertical projection of the same $GL(n, m, R)$-vector field.

Upon expansion of the vector field $GL(n, m, R)$, we have $\xi = \xi_1 \oplus \xi_2$. These vectors in the $\{E_i, e_\alpha\}$-frame have the following coordinates:

$$\xi_1 = \xi^i_1 \, E_i, \qquad \xi_2 = \xi^\alpha_2 \, e_\alpha,$$

that is,

$$\xi_1^i = \xi^i, \qquad \xi_1^\alpha = 0, \qquad \xi_2^i = 0, \qquad \xi_2^\alpha = \xi^\alpha + \Gamma_k^\alpha \xi^k,$$

and in the $\{e_i, e_\alpha\}$-frame, the coordinates

$$\xi_1^i = \xi^i, \qquad \xi_1^\alpha = -\Gamma_k^\alpha \xi^k, \qquad \xi_2^i = 0, \qquad \xi_2^\alpha = \xi^\alpha + \Gamma_k^\alpha \xi^k,$$

that is,

$$\xi_1 = \xi_1^i e_i - \xi_1^k \Gamma_k^\alpha e_\alpha, \qquad \xi_2 = \xi_2^\alpha e_\alpha.$$

If in the base of the space Lm(Vn) the GL(n, R)-vector field η^i is defined, then to that field there always uniquely corresponds the GL(n, m, R)-vector field defined on Lm(Vn). Such a correspondence is assigned as follows:

$$\xi^i = \eta^i, \qquad \xi^\alpha = -\eta^k \Gamma_k^\alpha. \tag{3}$$

Definition. The vector field ξ^A defined by equalities (3) is called a Γ-lift of the vector field η^i.

If T_B^A is the GL$(n, m, R) \times$ GL(n, m, R)-tensor field, then

$$T(\xi) = T_B^A \xi^B e_A \tag{4}$$

is an element of the space T_{n+m}. Since

$$T(\xi) = T_1(\xi) \oplus T_2(\xi),$$

therefore

$$T(\xi) = T_1(\xi_1) \oplus T_1(\xi_2) \oplus T_2(\xi_1) \oplus T_2(\xi_2). \tag{5}$$

Let

$$T_1(\xi_1) = a_j^i \xi_1^j E_i, \qquad T_1(\xi_2) = b_\beta^i E_i, \qquad T_2(\xi_1) = c_j^\alpha \xi_1^j e_\alpha, \qquad T_2(\xi_2) = d_\beta^\alpha \xi_2^\beta e_\alpha,$$

or

$$T_1(\xi_1) = a_j^i \xi^j e_i - a_j^i \xi^j \Gamma_i^\alpha e_\alpha, \qquad T_1(\xi_2) = b_\beta^i(\xi^\beta + \xi^k \Gamma_k^\beta)e_i - b_\beta^i(\xi^\beta + \xi^k \Gamma_k^\beta)\Gamma_i^\alpha e_\alpha,$$
$$T_2(\xi_1) = c_j^\alpha \xi^j e_\alpha, \qquad T_2(\xi_2) = d_\beta^\alpha(\xi^\beta + \xi^k \Gamma_k^\beta)e_\alpha.$$

Since these expansions take place for any vector field ξ^A, therefore, owing to equalities (4) and (5), we find that

$$T_j^i = a_j^i + b_\alpha^i \Gamma_j^\alpha, \qquad T_\alpha^i = b_\alpha^i, \qquad T_\beta^\alpha = d_\beta^\alpha - b_\beta^i \Gamma_i^\alpha, \qquad T_j^\alpha = c_j^\alpha - a_j^i \Gamma_i^\alpha - b_\beta^i \Gamma_j^\beta \Gamma_i^\alpha + d_\beta^\alpha \Gamma_j^\beta. \tag{6}$$

Obviously, the values a_j^i, b_α^i, c_j^α, d_β^α appearing in the above formulas, are the tensors. Thus, formulas (6) can be interpreted as a fully definite correspondence definable by the object of linear connectedness Γ_i^α.

Definition. The GL(n, m, R)-tensor field T_B^A defined by equality (6) is called a Γ-lift of an ordered quadruple of GL(n, R), GL(m, R), GL(n, m, R)-tensor fields a_j^i, b_α^i, c_j^α, d_β^α, defined on Lm(Vn).

2. Internal tensor structures

Definition. The space Lm(Vn) on which is defined the tensor field T_B^A satisfying the conditions

$$T_C^A T_B^C = \lambda \delta_B^A, \tag{7}$$

will be called a fiber space with a tensor structure.

If $\lambda = 0$, then the tensor structure will be called an almost dual structure, if $\lambda = -1$, it will be called an almost complex structure, and if $\lambda = 1$, it will be an almost binary structure [2].

We will focus our attention not on arbitrary tensor structures, but only on those which are generated by an object of linear connectedness and by certain vector fields. Such tensor structures will be called in the sequel internal tensor structures.

Differential equations of the tensor T^A_B have the form

$$\nabla T^A_B = \nabla_C T^A_B \omega^C \equiv \nabla_i T^A_B \omega^i + \nabla_\alpha T^A_B \widetilde{\theta^\alpha}. \tag{8}$$

Let

$$a^i_j = a\,\delta^i_j, \qquad b^i_\alpha = b\,\xi^i \eta_\alpha, \qquad c^\alpha_j = c\,\xi^\alpha \eta_j, \qquad d^\alpha_\beta = d\,\delta^\alpha_\beta,$$

where a, b, c, d are arbitrary scalars, η_α, η_j are the $\mathrm{GL}(m, R)$, $\mathrm{GL}(n, R)$-covector fields, and ξ^i, ξ^α are the $\mathrm{GL}(n, R)$, $\mathrm{GL}(m, R)$-vector fields. It is assumed that

$$\xi^i \eta_i = 1, \qquad \xi^\alpha \eta_\alpha = 1.$$

The lift of that quadruple of tensor fields has the form

$$\begin{aligned} &T^i_j = a\,\delta^i_j + b\,\xi^i \eta_\alpha \Gamma^\alpha_j, \quad T^i_\alpha = b\,\xi^i \eta_\alpha\,, \quad T^\alpha_\beta = d\,\delta^\alpha_\beta - b\,\xi^i \eta_\beta \Gamma^\alpha_i\,, \\ &T^\alpha_j = c\,\xi^\alpha \eta_j - a\,\Gamma^\alpha_j - b\,\xi^k \eta_\beta \Gamma^\beta_j \Gamma^\alpha_k + d\,\Gamma^\alpha_j\,. \end{aligned} \tag{9}$$

Written explicitly, the system of square equations (7) has the form

$$T^i_k T^k_j + T^i_\alpha T^\alpha_j = \lambda\,\delta^i_j, \qquad T^i_k T^k_\alpha + T^i_\beta T^\beta_\alpha = 0, \qquad T^\alpha_k T^k_j + T^\alpha_\beta T^\beta_j = 0, \qquad T^\alpha_k T^k_\beta + T^\alpha_\gamma T^\gamma_\beta = \lambda\,\delta^\alpha_\beta$$

whence it follows by virtue of (9) that the values are connected by the following relations:

$$\begin{aligned} &(a^2-\lambda)\delta^i_j + bc\,\xi^i \eta_j + b(a+d)\xi^i \eta_\gamma \Gamma^\gamma_j = 0, \qquad b(a+d)\xi^i \eta_\alpha = 0, \\ &(d^2-\lambda)\delta^\alpha_\beta + bc\,\xi^\alpha \eta_\beta - b(a+d)\xi^i \eta_\beta \Gamma^\alpha_i = 0, \\ &(d^2-a^2)\Gamma^\alpha_j + c(a+d)\xi^\alpha \eta_j - b(a+d)\xi^i \eta_\beta \Gamma^\alpha_i \Gamma^\beta_j + bc\,\xi^\alpha \eta_\gamma \Gamma^\gamma_j - bc\,\xi^k \eta_j \Gamma^\alpha_k = 0, \end{aligned}$$

that is,

$$\begin{aligned} &a^2 + bc - \lambda = 0, \qquad d^2 + bc - \lambda = 0, \qquad b(a+d) - 0, \\ &(d^2-a^2)\Gamma^\alpha_j + c(a+d)\xi^\alpha \eta_j + bc\,\xi^\alpha \eta_\gamma \Gamma^\gamma_j - bc\,\xi^k \eta_j \Gamma^\alpha_k = 0. \end{aligned}$$

We will seek only for those solutions which depend on a maximal number of parameters. If $b = 0$, $d + a = 0$, we obtain

$$d = -a, \qquad a^2 = \lambda.$$

In the other case, if $b \neq 0$, we obtain

$$d + a = 0, \qquad c\big(\xi^\alpha \eta_\gamma \Gamma^\gamma_j - \xi^k \eta_j \Gamma^\alpha_k\big) = 0.$$

This implies that $d = -a$, $c = 0$, $a^2 = \lambda$.

We have proved that there exist two two-parametric families of internal tensor structures (a, b, c are arbitrary parameters):

$$\begin{Vmatrix} a\,\delta^i_j & 0 \\ c\,\xi^\alpha \eta_j - 2a\,\Gamma^\alpha_j & -a\,\delta^\alpha_\beta \end{Vmatrix}, \tag{10}$$

$$\begin{Vmatrix} a\,\delta^i_j + b\,\xi^i \eta_\alpha \Gamma^\alpha_j & b\,\xi^i \eta_\alpha \\ -2a\,\Gamma^\alpha_j - b\,\xi^k \eta_\beta \Gamma^\beta_j \Gamma^\alpha_k & -a\,\delta^\alpha_\beta - b\,\xi^i \eta_\beta \Gamma^\alpha_i \end{Vmatrix}. \tag{11}$$

It should be, however, noted that each of families consists of different, in the main, tensor structures, since the first family of tensor structures consists of elements of the triangular algebra $\check{\mathfrak{J}}\,\mathrm{Lm(Vn)}$, and the second one consists of elements of the algebra $\mathfrak{J}(\mathrm{Lm(Vn)})$.

Thus, we have proved the following theorems.

Theorem 1. *If in the base* Vn *of space* Lm(Vn) *with linear connectedness* $\Gamma_i^\alpha(x, y)$ *there are the* $\mathrm{GL}(m, R)$*-vector field* $\xi^\alpha(x)$ *and the* $\mathrm{GL}(n, R)$*-covector field* $\eta_i(x)$*, then the tangent bundle of space* Lm(Vn) *has two one-parametric bundles of tensor structures which have no almost complex structures, but have only dual tensor structures and also structures of almost product.*

Theorem 2. *If in the space* Lm(Vn) *with linear connectedness* $\Gamma_i^\alpha(x, y)$ *there are the* $\mathrm{GL}(m, R)$*-vector field* $\xi^\alpha(x, y)$ *and the* $\mathrm{GL}(n, R)$*-covector field* $\eta_i(x, y)$*, then the tangent bundle of space* Lm(Vn) *has two one-parametric bundles of tensor structures which have no almost complex structures, but have only dual tensor structures and also structures of almost product.*

Theorem 3. *If in the base* Vn *of space* Lm(Vn) *with linear connectedness* $\Gamma_i^\alpha(x, y)$ *there are the* $\mathrm{GL}(n, R)$*-vector field* $\xi^i(x)$ *and the* $\mathrm{GL}(m, R)$*-covector field* $\eta_\alpha(x)$*, then the tangent bundle of space* Lm(Vn) *has two one-parametric bundles of tensor structures which have no almost complex structures, but have only dual tensor structures and also structures of almost product.*

Theorem 4. *If in the space* Lm(Vn) *with linear connectedness* $\Gamma_i^\alpha(x, y)$ *there are the* $\mathrm{GL}(n, R)$*-vector field* $\xi^i(x, y)$ *and the* $\mathrm{GL}(m, R)$*-covector field* $\eta_\alpha(x, y)$*, then the tangent bundle of space* Lm(Vn) *has two one-parametric bundles of tensor structures which have no almost complex structures, but have only dual tensor structures and also structures of almost product.*

3. The Nijenhuis tensor

If the tensor field T_B^A is defined by equations $\nabla T_B^A = \nabla_C T_B^A \omega^C$, then continuing these equations, we obtain

$$\nabla\left(\nabla_C T_B^A\right) - T_D^A \omega_{CB}^D + T_B^D \omega_{DC}^A = \nabla_D \nabla_C T_B^A \omega^D, \tag{12}$$

where

$$\nabla_{[D} \nabla_{C]} T_B^A = 0.$$

Rolling up Eq. (12) with T_E^C, we get

$$\nabla\left(T_E^C \nabla_C T_B^A\right) - T_E^C T_D^A \omega_{CB}^D + T_E^C T_B^D \omega_{DC}^A = 0.$$

This implies that

$$\nabla\left(T_B^C \nabla_C T_E^A\right) - T_D^A T_B^C \omega_{CE}^D + T_E^D T_B^C \omega_{DC}^A = 0,$$

and composing the difference, we obtain

$$\nabla\left(T_E^C \nabla_C T_B^A - T_B^C \nabla_C T_E^A\right) - T_E^C T_D^A \omega_{CB}^D + T_D^A T_B^C \omega_{CE}^D = 0.$$

Rolling up Eq. (12) with T_A^E, we find (after the change of indices) that

$$\nabla\left(T_C^A \nabla_E T_B^C\right) - T_D^C T_C^A \omega_{EB}^D + T_B^D T_C^A \omega_{DE}^C = 0,$$
$$\nabla\left(T_C^A \nabla_B T_E^C\right) - T_D^C T_C^A \omega_{EB}^D + T_E^D T_C^A \omega_{DB}^C = 0.$$

The last two equalities yield

$$\nabla\left(T_E^C \nabla_C T_B^A - T_B^C \nabla_C T_E^A - T_C^A \nabla_E T_B^C + T_C^A \nabla_B T_E^C\right) = 0,$$

and hence we obtain that the values

$$N_{EB}^A = T_E^C \nabla_C T_B^A - T_B^C \nabla_C T_E^A - T_C^A \nabla_E T_B^C + T_C^A \nabla_B T_E^C$$

form the tensor and we call it the Nijenhuis tensor.

Let us consider the Nijenhuis tensor N^A_{EB} of the internal tensor structure defined by formula (10).

Since the first part of Eq. (8) has the form $\nabla_i\, T^A_B\, \omega^i + \nabla_\alpha\, T^A_B\, \widetilde{\theta^\alpha}$, therefore, according to formula (11) and equalities

$$\begin{aligned}
&T^i_j = a\,\delta^i_j, \qquad T^i_\alpha = 0, \qquad T^\alpha_j = c\,\xi^\alpha\,\eta_j - 2a\,\Gamma^\alpha_j, \qquad T^\alpha_\beta = -a\,\delta^\alpha_\beta,\\
&\nabla_k\, T^i_j = 0, \qquad \nabla_\alpha\, T^i_j = 0, \qquad \nabla_k\, T^i_\alpha = 0, \qquad \nabla_\beta\, T^i_\alpha = 0, \qquad \nabla_k\, T^\alpha_\beta = 0, \qquad \nabla_\gamma\, T^\alpha_\beta = 0,\\
&\nabla_k\, T^\alpha_k = c\,\nabla_k(\xi^\alpha\,\eta_i) - 2a\,\nabla_k\,\Gamma^\alpha_i, \qquad \nabla_\beta\, T^\alpha_i = c\,\nabla_\beta(\xi^\alpha\,\eta_i) - 2a\,\nabla_\beta\,\Gamma^\alpha_i,
\end{aligned}$$

we obtain

$$\begin{aligned}
&N^i_{jk} = N^i_{j\alpha} = N^i_{\alpha j} = N^i_{\alpha\beta} = N^\alpha_{\beta i} = N^\alpha_{i\beta} = N^\alpha_{\beta\gamma} = 0,\\
&N^\alpha_{ij} = 4a^2\, R^\alpha_{ij} + 2ac\big(D_j(\xi^\alpha\,\eta_i) - D_i(\xi^\alpha\,\eta_j)\big) + c^2\,\xi^\gamma\,\eta_j\,\nabla_\gamma(\xi^\alpha\,\eta_i) - c^2\,\xi^\gamma\,\eta_i\,\nabla_\gamma(\xi^\alpha\,\eta),
\end{aligned}$$

where R^α_{ij} is the curvature tensor of connectedness Γ^α_i, and D_j is a nonholonomic covariant product of the first kind [3].

Thus we have the following theorems.

Theorem 5. *If on the base* Vn *of space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula* (10)*, then these structures are integrable, if and only if linear connectedness Γ^α_i is plane $R^\alpha_{ij} = 0$, and the vector field ξ^α and covector field η_i are co-constant (the covariant derivative of the first kind is equal to zero).*

Theorem 6. *If on the space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula* (10)*, then these structures are integrable, if and only if linear connectedness Γ^α_i is plane $R^\alpha_{ij} = 0$, and the vector field ξ^α and covector field η_i are co-constant (the covariant derivative of the first kind is equal to zero).*

Consider the Nijenhuis tensor N^A_{EB} of the internal tensor structure defined by formula (11). In this case we obtain

$$\begin{aligned}
N^i_{jk} &= 4ab\,\xi^i\eta_\alpha\,R^\alpha_{jk} + b^2\,\xi^p\,\eta_\beta\,\Gamma^\beta_j\,\Gamma^\gamma_k\,D_p(\xi^i\,\eta_\gamma) - b^2\,\xi^p\,\eta_\beta\,\Gamma^\gamma_j\,\Gamma^\beta_k\,D_p(\xi^i\,\eta_\gamma)\\
&\quad + b^2\,\xi^i\,\eta_\alpha\,\xi^p\,\eta_\beta\,\Gamma^\beta_j\,R^\alpha_{pk} - b^2\,\xi^i\,\eta_\beta\,\eta_\alpha\,\xi^p\,\Gamma^\beta_k\,R^\alpha_{pj},\\
N^i_{j\alpha} &= b^2\,\xi^p\,\eta_\alpha\,\xi^i\,\eta_\gamma\,R^\gamma_{pj} + b^2\,\xi^p\,\eta_\beta\,\Gamma^\beta_j\,D_p(\xi^i\,\eta_\alpha) - b^2\,\xi^p\,\eta_\alpha\,\Gamma^\beta_j\,D_p(\xi^i\,\eta_\beta),\\
N^i_{\alpha\beta} &= b^2\,\xi^p\,\eta_\alpha\,D_p(\xi^i\,\eta_\beta) - b^2\,\xi^p\,\eta_\beta\,D_p(\xi^i\,\eta_\alpha),\\
N^\alpha_{\beta\gamma} &= b^2\,\xi^p\,\eta_\gamma\,\Gamma^\alpha_i\,D_p(\xi^i\,\eta_\beta) - b^2\,\xi^p\,\eta_\beta\,\Gamma^\alpha_i\,D_p(\xi^i\,\eta_\gamma) + 2ab\,\Gamma^\alpha_p\nabla_\beta(\xi^p\,\eta_\gamma) - 2ab\,\Gamma^\alpha_p\nabla_\gamma(\xi^p\,\eta_\beta),\\
N^\alpha_{\beta i} &= 4ab\,\xi^p\eta_\beta\,R^\alpha_{pi} + b^2\,\xi^k\,\eta_\gamma\,\xi^p\,\eta_\beta\,\Gamma^\beta_j\,R^\gamma_{pi} + b^2\,\xi^p\,\eta_\gamma\,\Gamma^\gamma_i\,\Gamma^\alpha_k\,D_p(\xi^k\,\eta_\beta) - b^2\,\xi^p\,\eta_\beta\,\Gamma^\gamma_i\,\Gamma^\alpha_k\,D_p(\xi^k\,\eta_\gamma)\\
&\quad + 2ab\,\Gamma^\alpha_p\Gamma^\delta_i\,\nabla_\beta(\xi^p\,\eta_\delta) - 2ab\,\Gamma^\alpha_p\,\Gamma^\delta_i\,\nabla_\delta(\xi^p\,\eta_\beta),\\
N^\alpha_{ij} &= 4a^2\,R^\alpha_{ij} - 2ab\,\Gamma^\gamma_i\,\Gamma^\alpha_p\,R^p_{i\gamma} + 2ab\,\Gamma^\gamma_j\,\Gamma^\alpha_p\,R^p_{i\gamma} + b^2\,\xi^k\,\eta_\gamma\,\xi^p\,\eta_\delta\big(\Gamma^\gamma_i\,\Gamma^\delta_j - \Gamma^\gamma_j\,\Gamma^\delta_i\big)R^\alpha_{pk}\\
&\quad + b^2\,\xi^p\,\eta_\gamma\big(\Gamma^\gamma_i\,\Gamma^\delta_j - \Gamma^\gamma_j\,\Gamma^\delta_i\big)D_p(\xi^k\,\eta_\delta) + b^2\,\xi^k\,\eta_\gamma\,\xi^p\,\eta_\delta\,\Gamma^\alpha_k\big(\Gamma^\gamma_i\,R^\delta_{pj} - \Gamma^\gamma_j\,R^\delta_{pi}\big).
\end{aligned}$$

This results in the following theorems.

Theorem 7. *If in the base* Vn *of space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula* (11)*, then these structures are integrable, if and only if linear connectedness Γ^α_i is plane $R^\alpha_{ij} = 0$, and the vector field ξ^i and covector field η_α are co-constant (the covariant derivative of the first kind is equal to zero).*

Theorem 8. *If on the space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula* (11)*, then these structures are integrable, if and only if linear connectedness Γ^α_i is plane $R^\alpha_{ij} = 0$, and the vector field ξ^i and covector field η_α are co-constant (the covariant derivative of the first kind is equal to zero).*

Theorem 9. *If in the base* Vn *of space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula* (11)*, then these structures are not fully integrable, but the subobjects $\{N^i_{\alpha\beta}\}$, $\{N^i_{\alpha\beta}, N^\alpha_{\beta\gamma}\}$ of the Nijenhuis tensor vanish, when the vector field $\xi^i(x)$ and covector field $\eta_\alpha(x)$ are co-constant (the covariant derivative of the first kind is equal to zero).*

Theorem 10. *If in the space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula (11), then these structures are not fully integrable, but the subobject* $\{N^i_{\alpha\beta}\}$ *of the Nijenhuis tensor vanishes, when the vector field* $\xi^i(x, y)$ *and covector field* $\eta_\alpha(x, y)$ *are co-constant (the covariant derivative of the first kind is equal to zero).*

Theorem 11. *If on the space* Lm(Vn) *there are dual tensor structures and structures of almost product defined by formula (11), then these structures are not fully integrable, but the subobjects* $\{N^i_{\alpha\beta}\}$, $\{N^i_{j\alpha}, N^i_{\alpha\beta}\}$, $\{N^i_{jk}, N^i_{j\alpha}, N^i_{\alpha\beta}\}$, $\{N^i_{\alpha\beta}, N^\alpha_{ij}, N^i_{jk}, N^i_{j\alpha}, N^\alpha_{\beta\gamma}, N^\alpha_{\beta i}\}$ *of the Nijenhuis tensor vanish, when linear connectedness* Γ^α_i *is plane* $R^\alpha_{ij} = 0$, *and the vector field* $\xi^i(x, y)$ *and covector field* $\eta_\alpha(x, y)$ *are co-constant (the covariant derivative of the firs kind is equal to zero).*

4. *F*-structures

The tensor structure T^A_B is called F-structure if

$$T^A_B\, T^B_C\, T^C_D + \lambda\, T^A_D = 0 \quad (\lambda = \pm 1).$$

Written explicitly, this system has the form

$$\begin{aligned}
T^i_k\, T^k_p\, T^p_j + T^i_\gamma\, T^\gamma_k\, T^k_j + T^i_k\, T^k_\gamma\, T^\gamma_j + T^i_\gamma\, T^\gamma_\beta\, T^\beta_j + \lambda\, T^i_j &= 0,\\
T^\alpha_k\, T^k_p\, T^p_j + T^\alpha_k\, T^k_\gamma\, T^\gamma_j + T^\alpha_\gamma\, T^\gamma_k\, T^k_j + T^\alpha_\gamma\, T^\gamma_\beta\, T^\beta_j + \lambda\, T^\alpha_j &= 0,\\
T^i_k\, T^k_p\, T^p_\alpha + T^i_k\, T^k_\gamma\, T^\gamma_\alpha + T^i_\gamma\, T^\gamma_p\, T^p_\alpha + T^i_\gamma\, T^\gamma_\beta\, T^\beta_\alpha + \lambda\, T^i_\alpha &= 0,\\
T^\alpha_k\, T^k_p\, T^p_\beta + T^\alpha_k\, T^k_\gamma\, T^\gamma_\beta + T^\alpha_\gamma\, T^\gamma_p\, T^p_\beta + T^\alpha_\gamma\, T^\gamma_\delta\, T^\delta_\beta + \lambda\, T^\alpha_\beta &= 0.
\end{aligned}$$

From the above system and from equality (9) follows

$$\begin{aligned}
& b(a^2 + ad + bc + d^2 + \lambda)\xi^i\, \eta_\beta = 0,\\
& (a^3 + \lambda\, a)\delta^i_j + (a^2 b + b^2 c + bd^2 + abd + \eta b)\xi^i\, \eta_\alpha\, \Gamma^\alpha_j + (2abc + bcd)\xi^i\, \eta_j = 0,\\
& (d^3 + \lambda\, d)\delta^\alpha_\beta - (a^2 b + b^2 c + bd^2 + abd + \eta b)\xi^i\, \eta_\beta\, \Gamma^\alpha_i + (abc + 2bcd)\xi^\alpha\, \eta_\beta = 0,\\
& (d - a)(a^2 + ad + d^2 + \lambda)\Gamma^\alpha_j - bc(d + 2a)\xi^i\, \eta_j\, \Gamma^\alpha_i - b(a^2 + ad + bc + d^2 + \lambda)\xi^i\, \eta_\gamma\, \Gamma^\alpha_i\, \Gamma^\alpha_j\\
& \quad + (a^2 c + bc^2 + cd^2 + acd + \lambda c)\xi^\alpha\, \eta_j = 0,
\end{aligned}$$

or

$$\begin{aligned}
& b(a^2 + ad + bc + d^2 + \lambda) = 0, \qquad a^3 + 2abc + bcd + \lambda a = 0, \qquad d^3 + 2bcd + abc + \lambda d = 0,\\
& (d - a)(a^2 + ad + d^2 + \lambda)\Gamma^\alpha_j + c(a^2 + ad + bc + d^2 + \lambda)\xi^\alpha \eta_j\\
& \quad + bc(2d + a)\xi^\alpha \eta_\gamma \Gamma^\gamma_j - bc(d + 2a)\xi^i \eta_j \Gamma^\alpha_i = 0.
\end{aligned}$$

The second and third equalities result in

$$(d - a)(a^2 + ad + d^2 + bc + \lambda) = 0.$$

If $b = 0$, $d = a$, $a^2 + ad + d^2 + bc + \lambda \neq 0$, we obtain $c = 0$ and $a^2 + \lambda = 0$. Analogous results are obtained in the other cases, i.e., the real F-structures exist, if and only if $\lambda = -1$.

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